FINITE DIFFERENCE SCHEMES FOR MULTIDIMENSIONAL PARABOLIC TYPE EQUATION WITH CONSTANT COEFFICIENTS

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Summary

The paper deals with a simple unifed algorithm for construction of absolutely stable economical schemes to solve multidimensional parabolic type equations, where each difference equation completely approximates the given differencial equation. It is wortly to note that for the first time the constructed schemes are dependent on the dimension **p** only (they are not dependent on the weight).

Key words: Scheme. Stable. Approximate. Absolute.

Introduction

It is well-known that for multidimensional partial differential equation problems, the issue of construction of unified economical algoritms is related with considerable difficulties. Nevertheless, today exist numerous finite difference schemes of which one can single out those obtained by the upper layer factorization of the operator, those obtain by the method of summable approximation, etc.

The finite difference schemes which are now known as "alternative direction sweep" method was suggested first in 1955 simultaneously by Peaceman, Rachford and Douglas in [1,2]. These papers have become a basis to develop absolutely stable schemes (the so called economical schemes). Afterwards the schemes have been extended and deepened by Douqlas, Rachford, Baker, Oliphant, Brian, Samarshi, Yanenko, Marchuk, Gordeziani and others. We refer the reader to papers and the bibliography therein [3-5].

One of the peculiarities of the proposed algorithm is that each of the difference equations completely approximates a given differential equation. The latter enables us to define uniquely the boundary conditions ou the grid. Concerning this fact in comparison with other schemes see [5, §2,9].

It is well-known that in general difficulties occered in construction of simple (economical) absolutely stable schemes can not be avoided in the framework of schemes with homogeneous and simple approximation.. When the integration from a step to step is homogeneous, the structure of the difference scheme is to be changed. The latter complicates the approximation . In our case these two related problems are reduced just to the choice of the structure of the finite difference scheme, without causing any complications in approximation. Besides the proposed method allows to write down new schemes for an arbitrary dimension **p**. In particular, when **p=1, p=2** and the right-hand side of the equation vanishes, we get respectively the Krank-Nikolski and Douglas-Rachford schemes (see[1-5]).

For simplicity cousider the case of constant coefficients. Although with minor changes everything below can be stated for the case of a general second order equation with non-constant coefficients.

1. Setting of the problem, variation problem and difference schemes

Consider the first initial-boundary problem for a **p**-dimensional heat conductivity equation, where

$$\frac{\partial u}{\partial t} = Lu + f, \quad Lu = \sum_{i=1}^{p} L_i u, \quad L_i u = \frac{\partial^2 u}{\partial x_i^2}, \quad x \in G, \quad t \in (0,T] \tag{1.1}$$

$$u|_\Gamma = 0, \quad u(x,0) = u_0(x) \tag{1.2}$$

Let **G=Gop** be **p**-dimensional cube $\mathbf{0 \le x_i \le 1}$, $i = \overline{1,p}$. $\overline{\omega}_h = \{(i_1 h_1, \ldots, i_p h_p) \in G\}$

be the cube type net with the step **h** with respect to the variable $\mathbf{x_i}$, $h = \dfrac{1}{N_1}$, $\overline{\omega}_\tau$ -be the net with the step

$\tau = \dfrac{T}{N_0}$ on the interval $\mathbf{0 \le t \le T}$.

Let us consider first the algoritm of the construction of schemes for **p=1,2.** After established the rule we can write it in general case of arbitrary **p.**

To this end we rewrite equation (1.1) in the form

$$\beta\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}=\left(\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial^2 u}{\partial x_1^2}\right)+\beta f, \tag{1.3}$$

where β is a so far, not yet defined nonzero real number. In the sequel, both the stability and approximation of finite difference schemes will be dependent just on the selection of β and weight α.

In order to define $u(x_1,t)$, instead of problems (1.3) - (1.2), let us apply the well-known Hamilton principle:

$$I(u)=\int_{t_0}^{t_n}\left\{\int_0^1\left[\beta u\frac{\partial u}{\partial t}+\left(\frac{\partial u}{\partial t}\right)^2+\frac{1}{2n_1}\sum_{i=1}^{2n_1}\left[A_1^{(i)}\left(\frac{\partial u}{\partial \ell_i}\right)^2+B_1^{(i)}\left(\frac{\partial u}{\partial \ell_{i+1}}\right)^2+\right.\right.\right.$$

$$\left.\left.\left.+C_1^{(i)}\frac{\partial u}{\partial \ell_i}\frac{\partial u}{\partial \ell_{i+1}}\right]-2\beta fu\right]dx_1\right\}dt\to\min \tag{1.4}$$

I.e. if the function $u(x_1,t)$ is the solution to the given problem, then it provides the minimal value for the functional $I(u)$ in any time interval tn –t0 (t0=0).

If we devide the time interval by the small tj –tj-1, $j=\overline{1,n}$, than we can rewrite the condition (1.4) in the following form

$$I(u)=\sum_{j=1}^{n}I_j(u)\to\min$$

$$I_j(u)=\int_{t_{j-1}}^{t_j}\left\{\int_0^1\left[\beta u\frac{\partial u}{\partial t}+\left(\frac{\partial u}{\partial t}\right)^2+\frac{1}{2n_1}\sum_{i=1}^{2n_1}\left[A_1^{(i)}\left(\frac{\partial u}{\partial \ell_i}\right)^2+B_1^{(i)}\left(\frac{\partial u}{\partial \ell_{i+1}}\right)^2+C_1^{(i)}\frac{\partial u}{\partial \ell_i}\frac{\partial u}{\partial \ell_{i+1}}\right]-2\beta fu\right]dx_1\right\}dt$$

where

$$A_1^{(i)}=\frac{\cos^2\alpha_{i+1}+\beta\sin^2\alpha_{i+1}}{\sin^2(\alpha_{i+1}-\alpha_i)},\qquad B_1^{(i)}=\frac{\cos^2\alpha_i+\beta\sin^2\alpha_i}{\sin^2(\alpha_{i+1}-\alpha_i)},$$

$$C_1^{(i)}=\frac{-2(\cos\alpha_i\cos\alpha_{i+1}+\beta\sin\alpha_i\sin\alpha_{i+1})}{\sin^2(\alpha_{i+1}-\alpha_i)}$$

We note, that functional (1.4) is the identical representastion of the following functional

$$I(u)=\int_{t_0}^{t_n}\left\{\int_0^1\left[u\frac{\partial u}{\partial t}-\left(\frac{\partial u}{\partial x_1}\right)^2-2fu\right]dx_1\right\}dt$$

The directions $\overrightarrow{\ell_i}$ are defined by means of the angles α_i as follows :

$$\vec{\ell}=(\ell_1,\ell_2,\dots,\ell_{2n_1})\Rightarrow(\alpha_1,\alpha_2,\dots,\alpha_{2n_1}),\qquad \alpha_{n_1+i}=\pi+\alpha_i,\qquad \vec{\ell}_i=-\vec{\ell}_{n_1+i},\qquad i=\overline{1,n_1}$$

Let us consider one of the above possibilities. Namely when in the elementary cell containing a basis knot (x_i, t_j) (it is a rectangle with the centroid (xi,tj) the four directions ℓ_i, $i=\overline{1,4}$ **(n_1=2)** get out from the point (x_i,t_j), where

$\alpha_1=\pi-\alpha_2$, $\alpha_3-\alpha_2=2\alpha_1$, $\alpha_4-\alpha_3=\pi-2\alpha_1$, $\alpha_1'-\alpha_4=2\alpha_1$, $\alpha_1'=2\pi+\alpha_1$.

Here α_1 is the counterclockwise angle between the axis ox_1 and the direction ℓ_1 .

Let us exploit the mean value formula and write down the difference functional (with respect to basis knot (x_i,t_j) and to the fictitious knots $(x_i+\frac{h}{2}, t_j+\frac{\tau}{2})$, $(x_i-\frac{h}{2}, t_j+\frac{\tau}{2})$, $(x_i-\frac{h}{2}, t_j-\frac{\tau}{2})$, $(x_i+\frac{h}{2}, t_j-\frac{\tau}{2})$, where **τ** is the net step along the **T** axis, $t_j = \mathbf{j\tau}$, j=0,1,...; while **h** is the step along the ox_1 axis) corresponding to the functional (1.4). We call the knots to the fictitius, since the values of a function in the finite difference schemes are not involved because they are cancelled.

In the sequel we leave the same notation **U** for the net function

$$\begin{aligned} I_h(u) = \Bigg\{ & \left[\frac{\beta}{2}u(u_t + u_{\bar t}) + \left(u_t^2 + u_{\bar t}^2\right)\right]_{(x_i,t_j)} + \frac{1}{4}\sum_{i=1}^{4}\Big[A_1^{(i)}u_{\ell_i}^2 + B_1^{(i)}u_{\ell_{i+1}}^2 + \\ & + C_1^{(i)}u_{\ell_i}u_{\ell_{i+1}}\Big]_{(x_i,t_j)} + \frac{1}{4}\left(\sum_{i=1}^{4}C_1^{(i)}u_{\ell_i}u_{\ell_{i+1}}\right)_{\left(x_i+\frac{h}{2},t_j+\frac{\tau}{2}\right),\ldots,\left(x_i+\frac{h}{2},t_j-\frac{\tau}{2}\right)}\Bigg\} S_0, \end{aligned} \tag{1.5}$$

where **S0=4hτ,** is the area of elementary cell. We use the folloving notations :

$$\left(u_{\ell_1}\right)_{(x_i,t_j)} = \frac{u(x_i+h,t_j+\tau)-u(x_i,t_j)}{\sqrt{h^2+\tau^2}}, \quad \left(u_{\bar\ell_1}\right)_{(x_i,t_j)} = \frac{u(x_i,t_j)-u(x_i-h,t_j-\tau)}{\sqrt{h^2+\tau^2}}$$

$$u_{x_i} = \frac{u(x_i+h,t_j)-u(x_i,t_j)}{h}, \qquad u_{\bar x_i} = \frac{u(x_i,t_j)-u(x_i-h,t_j)}{h},$$

$$u_{\overset{0}{x}} = \frac{u_x+u_{\bar x}}{2h}, \qquad u_t = \frac{u(x_i,t_j+\tau)-u(x_i,t_j)}{\tau} = \frac{\hat u - u}{\tau},$$

$$u_{\bar t} = \frac{u(x_i,t_j)-u(x_i,t_j-\tau)}{\tau} = \frac{u-\breve u}{\tau}, \quad u_{\overset{0}{t}} = \frac{\hat u - \breve u}{2\tau}.$$

Inserting the values of coefficients $A_1^{(i)}$, $B_1^{(i)}$, $C_1^{(i)}$ in to functional (1.5) and writing it in a expanded form, we obtain

$$I_h(u)=\left\{\left[\frac{\beta}{2}u(u_t+u_{\bar t})+\left(u_t^2+u_{\bar t}^2\right)\right]_{(x_i,t_j)}+\frac{1}{4}\cdot\frac{\cos^2\alpha_1+\beta\sin^2\alpha_1}{\sin^2(\alpha_{i+1}-\alpha_i)}\cdot\left[\left(u_{\ell_1}^2+u_{\ell_2}^2+\right.\right.\right.$$

$$+\frac{2\left(\cos^2\alpha_1-\beta\sin^2\alpha_1\right)}{\cos^2\alpha_1+\beta\sin^2\alpha_1}u_{\ell_1}u_{\ell_2}\Bigg)+\Bigg(u_{\ell_2}^2+u_{\ell_3}^2+\frac{2\left(\cos^2\alpha_1-\beta\sin^2\alpha_1\right)}{\cos^2\alpha_1+\beta\sin^2\alpha_1}u_{\ell_2}u_{\ell_3}\Bigg)+\Bigg(u_{\ell_3}^2+u_{\ell_4}^2+$$

$$+\frac{2\left(\cos^2\alpha_1-\beta\sin^2\alpha_1\right)}{\cos^2\alpha_1+\beta\sin^2\alpha_1}u_{\ell_3}u_{\ell_4}\Bigg)+\Bigg(u_{\ell_1}^2+u_{\ell_4}^2+\frac{2\left(\cos^2\alpha_1-\beta\sin^2\alpha_1\right)}{\cos^2\alpha_1+\beta\sin^2\alpha_1}u_{\ell_1}u_{\ell_4}\Bigg)\Bigg]_{(x_i,t_j)}+$$

$$+2\frac{\cos^2\alpha_1-\beta\sin^2\alpha_1}{\cos^2\alpha_1+\beta\sin^2\alpha_1}(u_{\ell_3}u_{\ell_4}-u_{\ell_2}u_{\ell_3})_{\left(x_i+\frac{h}{2},t_j+\frac{\tau}{2}\right)}+$$

$$+2\frac{\cos^2\alpha_1-\beta\sin^2\alpha_1}{\cos^2\alpha_1+\beta\sin^2\alpha_1}(u_{\ell_3}u_{\ell_4}-u_{\ell_1}u_{\ell_4})_{\left(x_i-\frac{h}{2},t_j+\frac{\tau}{2}\right)}+$$

$$+2\frac{\cos^2\alpha_1-\beta\sin^2\alpha_1}{\cos^2\alpha_1+\beta\sin^2\alpha_1}(u_{\ell_1}u_{\ell_2}-u_{\ell_1}u_{\ell_4})_{\left(x_i-\frac{h}{2},t_j-\frac{\tau}{2}\right)}+$$

$$\left.+2\frac{\cos^2\alpha_1-\beta\sin^2\alpha_1}{\cos^2\alpha_1+\beta\sin^2\alpha_1}(u_{\ell_1}u_{\ell_2}-u_{\ell_2}u_{\ell_3})_{\left(x_i+\frac{h}{2},t_j-\frac{\tau}{2}\right)}-2\beta(fu)_{(x_i,t_j)}\right\}S_0.$$

Here are taken into account the follouving equations :

$$\alpha_2-\alpha_1=\pi-2\alpha_1,\quad \alpha_3-\alpha_2=2\alpha_1,\ \alpha_4-\alpha_3=\pi-2\alpha_1,\quad \alpha_1'-\alpha_4=2\alpha_1,\ \alpha_1'=2\pi+\alpha_1$$

$$\sin^2(\alpha_{i+1}-\alpha_i)=\sin^2 2\alpha_1\quad,\ i=\overline{1,4};$$

$$(\alpha_2=\pi-\alpha_1,\ \alpha_3=\pi+\alpha_1\ ,\ \alpha_4=2\pi-\alpha_1)$$

If we insert the values of difference operators in the obtained functional, use the Hamilton principle and carry out elementary but routine work, then we obtain the following difference scheme with respect to the basic knot point (x_i,t_j)

$$u_{\overset{0}{t}}+\frac{h^2+\beta\tau^2}{2\beta h^2}u_{t\bar t}=\frac{\left(h^2+\tau^2\right)\left(h^2+\beta\tau^2\right)}{\beta\cdot(2\tau h)^2}(u_{\ell_1\bar\ell_1}+u_{\ell_2\bar\ell_2})-\frac{h^2-\beta\tau^2}{2\beta\tau^2}u_{x_1\bar x_1}+f_{ij},$$

After a few transformations we can rewrite it in the following cannonical form

$$u_{\overset{0}{t}}+\tau^2Ru_{t\bar t}=u_{x_1\bar x_1}+f\quad,\quad (f=f(x_i,t_j)=f_{ij})\ , \tag{1.6}$$

where

$$R=\frac{h^2+\beta\tau^2}{4\beta\tau^2}L,\qquad L=-\Delta_{11},\qquad \Delta_{11}u=u_{x_1\bar x_1}.$$

If in scheme (1.6) we replace **β** by $\beta=\frac{1}{\sigma^*}\frac{h^2}{\tau^2}$, then we get the equivalent to (1.6) scheme, which we call **σ**-parametric basis scheme

$$u_{\overset{0}{t}}+\sigma\tau^2Ru_{t\bar t}=u_{x_1\bar x_1}+f\quad, \tag{1.7}$$

where $\sigma=\frac{1+\sigma^*}{4}>0$, $\left(\sigma\neq\frac{1}{4}\right)$.

2. Finite difference schemes for the equation $\frac{\partial u}{\partial t}=\sum_{i=1}^{2}\frac{\partial^2 u}{\partial x_i^2}+f$.

By analogy with the derivation of the scheme (1.7) let us consider the following identical representations for the given equation

$$\beta\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}=\left(\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial^2 u}{\partial x_1^2}\right)_{(\delta_1)}+\beta\frac{\partial^2 u}{\partial x_2^2}+\beta f\ , \tag{2.1}$$

$$\beta\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial t^2}=\left(\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial^2 u}{\partial x_2^2}\right)_{(\delta_2)}+\beta\frac{\partial^2 u}{\partial x_1^2}+\beta f\ , \tag{2.2}$$

$$\beta\frac{\partial u}{\partial t}+2\frac{\partial^2 u}{\partial t^2}=\left(\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial^2 u}{\partial x_1^2}\right)_{(\delta_1)}+\left(\frac{\partial^2 u}{\partial t^2}+\beta\frac{\partial^2 u}{\partial x_2^2}\right)_{(\delta_2)}+\beta f\ . \tag{2.3}$$

Similarly as in one-dimensional case we use the mean value formula and write down the difference functionals corresponding to the equations (2.1) - (2.3) in the elementary cell of dimensions (**2h, 2h , 2τ**) with respect to the centroid **(x_{1i}, x_{2j}, t_k)**,

$$I_{1h}(u)=\left\{\left[\frac{\beta}{2}u(u_t+u_{\bar t})+\left(u_t^2+u_{\bar t}^2\right)\right]_{(x_{1i},x_{2j},t_k)}+\frac{1}{4}\sum_{i=1}^{4}\left[\overset{(i)}{A_1}u^2_{\ell_i}+\overset{(i)}{B_1}u^2_{\ell_{i+1}}+\right.\right. \tag{2.4}$$
$$\left.\left.+\overset{(i)}{C_1}u_{\ell_i}u_{\ell_{i+1}}\right]_{(\delta_1)}-\beta\left(u^2_{x_2}+u^2_{\bar x_2}\right)_{(x_{1i},x_{2j},t_k)}-2\beta(fu)_{(x_{1i},x_{2j},t_k)}\right\}V_0$$

$$I_{2h}(u)=\left\{\left[\frac{\beta}{2}u(u_t+u_{\bar t})+\left(u_t^2+u_{\bar t}^2\right)\right]_{(x_{1i},x_{2j},t_k)}+\frac{1}{4}\sum_{i=1}^{4}\left[\overset{(i)}{A_1}u^2_{\ell_i}+\overset{(i)}{B_1}u^2_{\ell_{i+1}}+\right.\right. \tag{2.5}$$
$$\left.\left.+\overset{(i)}{C_1}u_{\ell_i}u_{\ell_{i+1}}\right]_{(\delta_2)}-\beta\left(u^2_{x_1}+u^2_{\bar x_1}\right)_{(x_{1i},x_{2j},t_k)}-2\beta(fu)_{(x_{1i},x_{2j},t_k)}\right\}V_0$$

$$I_{3h}(u)=\left\{\left[\frac{\beta}{2}u(u_t+u_{\bar t})+2\left(u_t^2+u_{\bar t}^2\right)\right]_{(x_{1i},x_{2j},t_k)}+\frac{1}{4}\sum_{k=1}^{2}\sum_{i=1}^{4}\left[\overset{(i)}{A_k}u^2_{\ell_i}+\overset{(i)}{B_k}u^2_{\ell_{i+1}}+\right.\right. \tag{2.6}$$
$$\left.\left.+\overset{(i)}{C_k}u_{\ell_i}u_{\ell_{i+1}}\right]_{(\delta_k)}-2\beta(fu)_{(x_{1i},x_{2j},t_k)}\right\}V_0$$

where v_0 is the volume of on elementary call. The indices **δ_1** and **δ_2** (in the sequal - **δ_3**) indicate the cut **δi** , along which the directions **(ℓ_1, ℓ_2, ℓ_3, ℓ_4)** are considered. In their turn **δi** represent the cuts parallel to the coordinate planes **x_1oT , x_2 o T , x_1 o x_2** passing through the centroid **(x_{1i}, x_{2j}, t_k)** of an elementary cell . Now we choose directions emanating from knot **(x_{1i}, x_{2j} , t_k)** on each cut as follows :

$$(\ell_1,\ \ell_2,\ \ell_3,\ \ell_4)_{(\delta i)} \Rightarrow (\alpha_1,\ \pi-\alpha_1,\ \pi+\alpha_1,\ 2\pi-\alpha_1)$$

For simplicity, we use the same notations for angles on each cut **δ_i** Moreover the values of a trigonometric function, say for argument **α_1** , are different e.g.

$$(\cos\alpha_1)_{\delta_1}=\frac{h_1}{\sqrt{h_1^2+\tau^2}}\ ,\ (\cos\alpha_1)_{(\delta_2)}=\frac{h_2}{\sqrt{h_2^2+\tau^2}}\ ,\ (\cos\alpha_1)_{(\delta_3)}=\frac{h_1}{\sqrt{h_1^2+h_2^2}}\ ,$$

(when **$h_1 \neq h_2$**)

According to the Hamilton principle, the difference functionals (2.4) – (2.6) define respectively the following three-layer finite difference schemes:

$$u_{\overset{0}{t}}+\sigma\tau^2R_1u_{t\bar t}=u_{x_1\bar x_1}+u_{x_2\bar x_2}+f\ ,\ i=\overline{1,N_1}\qquad \text{(for a fixed j)} \tag{2.7}$$

$$u_{\overset{0}{t}}+\sigma\tau^2R_2u_{t\bar t}=u_{x_1\bar x_1}+u_{x_2\bar x_2}+f\ ,\ j=\overline{1,N_2}\qquad \text{(for a fixed j)} \tag{2.8}$$

$$u_{\overset{0}{t}} + \sigma\tau^2(R_1 + R_2)u_{t\bar{t}} = u_{x_1\bar{x}_1} + u_{x_2\bar{x}_2} + f \quad , \qquad \mathbf{i,j}=\overline{1,N} \tag{2.9}$$

where $\mathbf{f = f_{ijk} = f(x_{1i}, x_{2j}, t_k)}$, $\mathbf{N_1}$ is the number of direction points along the $\mathbf{ox_1}$ axis , while $\mathbf{N_2}$ –along the $\mathbf{ox_2}$ axis, $\mathbf{N=(N_1,N_2)}$

$$R_i = -\Delta_{ii} \quad , \quad \Delta_{ii}u = u_{x_i}u_{\bar{x}_i} \quad , \quad i = 1,2 \ .$$

3. Finite difference schemes for the equation $\frac{\partial u}{\partial t} = \sum_{i=1}^{p} \frac{\partial^2 u}{\partial x_i^2} + f$.

For the given equation consider the following identical representations

$$\beta\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial^2 u}{\partial x_1^2}\right)_{(\delta_1)} + \beta\sum_{i=2}^{p}\frac{\partial^2 u}{\partial x_i^2} + \beta f \ ,$$

$$\beta\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial^2 u}{\partial x_2^2}\right)_{(\delta_2)} + \beta\sum_{\substack{i=1\\ i\neq 2}}^{p}\frac{\partial^2 u}{\partial x_i^2} + \beta f \ ,$$

$$\beta\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial^2 u}{\partial x_k^2}\right)_{(\delta k)} + \beta\sum_{\substack{i=1\\ i\neq k}}^{p}\frac{\partial^2 u}{\partial x_i^2} + \beta f,$$

$$\beta\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = \left(\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial^2 u}{\partial x_p^2}\right)_{(\delta p)} + \beta\sum_{i=1}^{p-1}\frac{\partial^2 u}{\partial x_i^2} + \beta f, \quad \left(k = \overline{1,p}\right) \text{ and}$$

$$\beta\frac{\partial u}{\partial t} + p\frac{\partial^2 u}{\partial t^2} = \sum_{i=1}^{p}\left(\frac{\partial^2 u}{\partial t^2} + \beta\frac{\partial^2 u}{\partial x_i^2}\right)_{(\delta i)} + \beta f.$$

where δ_i this time represent the hyperlanes.

If we repeat word for word the method of receiving the finite difference schemes, we obtain the following schemes

$$u_{\overset{0}{t}} + \sigma\tau^2 R_k u_{t\bar{t}} = \sum_{i=1}^{p} u_{x_i\bar{x}_i} + f \ , \quad k = \overline{1,p} \tag{3.1}$$

$$u_{\overset{0}{t}} + \sigma\tau^2\left(\sum_{i=1}^{p} R_i\right)u_{t\bar{t}} = \sum_{i=1}^{p} u_{x_i\bar{x}_i} + f \ . \tag{3.2}$$

Note that (3.1) represents **p** independent schemes. The fixed **k** determines the variable $\mathbf{x_k}$ with respect to which the “sweep” method is being used for finding solution **U**. As to the solution of problem by mean of finite difference scheme (3.2), we can apply the method of matrix factorization or some economical scheme, of the operator $R = \sum_{i=1}^{p} R_i$ is factorized on the upper layer (see [5, §2, p6]) or if we consider the analogy of the Baker-Oliphant scheme (see [6]).

4. Weight schemes and an algorithm for the computation of difference problem

Let us cousider the case with **p=2** and for schemes (2.7), (2.8) write down the following α weight schemes

$$\frac{u^{n+1} - u^{n-1}}{2\tau} - \sigma\tau^2\Delta_{11}u_{t\bar{t}} = \Delta_{11}\left(\alpha u^{n+1} + (1-\alpha)u^n\right) + \Delta_{22}u^{n-1} + f^n, \tag{4.1}$$

$$\frac{u^{n+2}-u^{n}}{2\tau}-\sigma\tau^{2}\Delta_{22}u_{t\bar{t}}=\Delta_{22}\left(\alpha u^{n+2}+(1-\alpha)u^{n+1}\right)+\Delta_{11}u^{n}+f^{n+1}, \tag{4.2}$$

First we write down the scheme (4.1) on the **(n -** $\frac{1}{2}$ **)** layer or for step $\frac{\tau}{2}$.

$$\frac{u^{n}-u^{n-1}}{\tau}-\sigma\left(\frac{\tau}{2}\right)^{2}\Delta_{11}u_{t\bar{t}}=\Delta_{11}\left(\alpha u^{n}+(1-\alpha)u^{n-1}\right)+\Delta_{22}u^{n-1}+f^{n-\frac{1}{2}}$$

and find the solution on the **n**-th layer. The scheme is three-layer, but now and in the sequel if the parametrers **σ, α** and the weight are chosen from the condition $\sigma=\frac{1-\alpha}{2}$,we come to the two-layer scheme, where only the right-hand side **f** depends on the half step $f^{n-\frac{1}{2}}=f\left(i_1h_1, i_2h_2, \frac{\tau}{2}\right)$.

Then, alternating the schemes (4.1) and (4.2) we obtain the solutions on the **n+1, . . . ,n+p** layers and so on.

When **p=3**, we consider for the scheme (3.1) the following schemes

$$\frac{u^{n+1}-u^{n-1}}{2\tau}-\frac{1-\alpha}{2}\Delta_{11}\left(u^{n+1}+u^{n-1}\right)=\alpha\Delta_{11}u^{n+1}+\Delta_{22}u^{n-1}+\Delta_{33}u^{n-1}+f^{n}, \tag{4.3}$$

$$\frac{u^{n+2}-u^{n}}{2\tau}-\frac{1-\alpha}{2}\Delta_{22}\left(u^{n+2}+u^{n}\right)=\alpha\Delta_{22}u^{n+2}+\Delta_{11}u^{n}+\Delta_{33}u^{n}+f^{n+1}, \tag{4.4}$$

$$\frac{u^{n+3}-u^{n+1}}{2\tau}-\frac{1-\alpha}{2}\Delta_{33}\left(u^{n+3}+u^{n+1}\right)=\alpha\Delta_{33}u^{n+3}+\Delta_{11}u^{n+1}+\Delta_{22}u^{n+1}+f^{n+2} \tag{4.5}$$

Analogously write down the scheme (4.3) for the **(n -** $\frac{1}{2}$ **)**-th layer :

$$\frac{u^{n}-u^{n-1}}{\tau}-\frac{1-\alpha}{2}\Delta_{11}\left(u^{n}+u^{n-1}\right)=\alpha\Delta_{11}u^{n}+\Delta_{22}u^{n-1}+\Delta_{33}u^{n-1}+f^{n-\frac{1}{2}} \tag{4.6}$$

After finding solutions by scheme (4.6) on **n**-th layer, we alternate the schemes (4.3), (4.4) and (4.5) to find solutions on the layer **n+1, . . . , n+p,** etc.

Having the rule described, we can write down the **α**-weight schemes by means of scheme (3.1) for a general **p**.

$$\frac{u^{n+k}-u^{n+k-2}}{2\tau}-\frac{1-\alpha}{2}\Delta_{kk}\left(u^{n+k}+u^{n+k-2}\right)=\alpha\Delta_{kk}u^{n+k}+\sum_{\substack{i=1\\ i\neq k}}^{p}\Delta_{ii}u^{n+k-2}+f^{n+k-1},\ k=\overline{1,p} \tag{4.7}$$

The solution **i**s u^{n+pk_1}, where $K_1=1,2,...,\frac{N_0-1}{p}$. **($N_0$ – 1** is a number divisible by **p**).

Every **k**=$\overline{1,p}$ two-layer scheme completely approximates the given difference equation . Hence by the **p**-time successive application of them we can define the solution to the difference problem on the **(n+pk)**-th layer. The latter means the use of “sweep” method along the axes **OX**$_1$, . . . , **OX**$_p$. So we obtain almost absolutely stable, having complete approximation schemes for the case **p ≥ 3** and with the exactness **O(** τ^2+|**h**|^2**)** by alternating the independently constructed schemes.

5. Study of the difference schemes

So far we consider mostly the methods of construction of schemes .Now we show the stability of the constructed schemes by use of methods of harmounic analysis.

Let us prove the unconditional stability of the scheme (4.7), when **p=1**. Suppose

$$u^n = \rho^n e^{i(k_1x_1+k_2x_2+\cdots+k_px_p)}\ ,\quad \rho^n = e^{\omega n\tau}\ ,\quad \left(i=\sqrt{-1}\right) \qquad (5.1)$$

Assuming that **f = 0** and substituting (5.1) into equation (4.7), we obtain the following dispersion equation

$$\left(1+\frac{1+\alpha}{2}a_1\right)\rho^2+\left(\frac{1-\alpha}{2}a_1-1\right)=0$$

For the coefficient of error increase we obtain

$$\rho=\sqrt{\frac{1-\frac{1-\alpha}{2}a_1}{1+\frac{1+\alpha}{2}a_1}}\ ,$$

where, in general $a_j = 4r_i\sin^2\frac{k_jh_j}{2}$, $r_j=\frac{2\tau}{h_j^2}$, $j=\overline{1,p}$.

For the stability it is necessary that $|\rho|\le 1$, i.e.

$$\frac{\left|1-\frac{1-\alpha}{2}a_1\right|}{1+\frac{1+\alpha}{2}a_1}\le 1 \qquad (5.2)$$

The ineguality (5.2) is valid for any **τ, h**, when **α ≥ 0.**

Let us consider the case with **p=2** . The dispersion equations have the folloving forms.

$$\left(1+\frac{1+\alpha}{2}a_1\right)\rho^2+\left(\frac{1-\alpha}{2}a_1+a_2-1\right)=0\ ,$$
$$\left(1+\frac{1+\alpha}{2}a_2\right)\rho^2+\left(\frac{1-\alpha}{2}a_2+a_1-1\right)=0$$

For the stability the following condition is required

$$|\rho|=|\rho_1\rho_2|\le 1 \qquad (5.3)$$

where

$$\rho_1=\sqrt{\frac{1-\frac{1-\alpha}{2}a_1-a_2}{1+\frac{1+\alpha}{2}a_1}}\ ,\quad \rho_2=\sqrt{\frac{1-\frac{1-\alpha}{2}a_2-a_1}{1+\frac{1+\alpha}{2}a_2}}$$

or equivalently

$$\frac{\left|1-\frac{1-\alpha}{2}a_1-a_2\right|}{1+\frac{1+\alpha}{2}a_1}\le 1\ ,\quad \frac{\left|1-\frac{1-\alpha}{2}a_2-a_1\right|}{1+\frac{1+\alpha}{2}a_2}\le 1\ .$$

Let us simplify the first inequality, then the second inequality could be simplified analogously

$$\begin{cases}\dfrac{1-\frac{1-\alpha}{2}a_1-a_2}{1+\frac{1+\alpha}{2}a_1}\le 1,\\[2ex] \dfrac{1-\frac{1-\alpha}{2}a_1-a_2}{1+\frac{1+\alpha}{2}a_1}\ge -1\end{cases}$$

Let us add $\dfrac{1-\alpha}{1+\alpha} \geq 0$ **($0 \leq \alpha \leq 1$)** to both sides of the inequality. Then we obtain that:

$$\begin{cases} \dfrac{\dfrac{2}{1+\alpha}-a_2}{1+\dfrac{1+\alpha}{2}a_1} \leq \dfrac{2}{1+\alpha} \\[2ex] \dfrac{a_2-\dfrac{2}{1+\alpha}}{1+\dfrac{1+\alpha}{2}a_1} \leq \dfrac{2\alpha}{1+\alpha} \end{cases}$$

Here the first iniquality is true, becauce $a_i > 0$. According the second inequality for the condition (5.3) it is necessary that

$$\frac{\left|a_2-\dfrac{2}{1+\alpha}\right|}{1+\dfrac{1+\alpha}{2}a_1}\cdot\frac{\left|a_1-\dfrac{2}{1+\alpha}\right|}{1+\dfrac{1+\alpha}{2}a_2} \leq \left(\frac{2\alpha}{1+\alpha}\right)^2 \tag{5.4}$$

Let us transpose the denominators (this means the alternating use of schemes). For the validity of (5.4) we require that the following system holds true

$$\begin{cases} \dfrac{\left|a_2-\dfrac{2}{1+\alpha}\right|}{1+\dfrac{1+\alpha}{2}a_2} \leq \dfrac{2\alpha}{1+\alpha} \\[2ex] \dfrac{\left|a_1-\dfrac{2}{1+\alpha}\right|}{1+\dfrac{1+\alpha}{2}a_1} \geq \dfrac{2\alpha}{1+\alpha} \end{cases} \Rightarrow \begin{cases} (1-\alpha)a_i \leq 2 \\[1ex] (1+\alpha)a_i \geq 2\dfrac{1-\alpha}{1+\alpha}, \quad i=1,2. \end{cases}$$

Or which is the same

$$2\frac{1-\alpha}{(1+\alpha)^2} \leq a_i \leq \frac{2}{1-\alpha}, \quad \textbf{if} \ \ \alpha \leq 1 \tag{5.5}$$

We require also

$$\frac{2}{1-\alpha} \geq 2\frac{1-\alpha}{(1+\alpha)^2}$$

Hence **$\alpha \geq 0$ or**, for **$p = 2$, $0 \leq \alpha \leq 1$.**

Remark:. For sufficiently large $\dfrac{\tau}{h_i^2}$ **i=1,2,...,**i.e. $a_1 = a_2$ **,** we obtain the following expansion for **ρ**

$$\rho = \sqrt{\left(\frac{1-\dfrac{3-\alpha}{2}a}{1+\dfrac{1+\alpha}{2}a}\right)^2} = \frac{\left|1-\dfrac{3-\alpha}{2}a\right|}{1+\dfrac{1+\alpha}{2}a} = \frac{\left|1-\dfrac{3-\alpha}{2}\cdot\dfrac{2}{1-\alpha}\right|}{1+\dfrac{1+\alpha}{2}\cdot\dfrac{2}{1-\alpha}} = 1 .$$

Let us consider the case with p**=3.**

According to (4.7), when **k=1,2,3**, the dispersion equations have the following forms

$$\left(1+\frac{1+\alpha}{2}a_1\right)\rho^2 + \left(\frac{1-\alpha}{2}a_1 + a_2 + a_3 - 1\right) = 0 \ ,$$

$$\left(1+\frac{1+\alpha}{2}a_2\right)\rho^2 + \left(\frac{1-\alpha}{2}a_2 + a_1 + a_3 - 1\right) = 0 \, ,$$

$$\left(1+\frac{1+\alpha}{2}a_3\right)\rho^2+\left(\frac{1-\alpha}{2}+a_1+a_2-1\right)=0\ ,$$

Analogously to (5.2), (5.3) we require that

$$|\rho|=|\rho_1\rho_2\rho_3|=\sqrt{\frac{\left|1-\frac{1-\alpha}{2}a_1-a_2-a_3\right|}{1+\frac{1+\alpha}{2}a_1}\cdot\frac{\left|1-\frac{1-\alpha}{2}a_2-a_1-a_3\right|}{1+\frac{1+\alpha}{2}a_2}\cdot\frac{\left|1-\frac{1-\alpha}{2}a_3-a_2-a_1\right|}{1+\frac{1+\alpha}{2}a_3}}\le 1 \qquad (5.6)$$

In order to satisfy the inequality (5.6), we require that in the expression under the root the first multiplier is less than 1 and analogously for other multipliers :

$$\frac{\left|1-\frac{1-\alpha}{2}a_1-a_2-a_3\right|}{1+\frac{1+\alpha}{2}a_1}\le 1$$

Let us simpify the inequality, remove the moduls and add $\frac{1-\alpha}{1+\alpha}$ to both sides of the inequality. We get that in order to satisfy the last inequality the follwing is sufficient

$$\frac{\left|a_2+a_3-\frac{2}{1+\alpha}\right|}{1+\frac{1+\alpha}{2}a_1}\le\frac{2\alpha}{1+\alpha} \qquad (5.7)$$

We do the same for **p** the remaining multipliers :

$$\frac{\left|a_1+a_3-\frac{2}{1+\alpha}\right|}{1+\frac{1+\alpha}{2}a_2}\le\frac{2\alpha}{1+\alpha} \qquad (5.8)$$

$$\frac{\left|a_1+a_2-\frac{2}{1+\alpha}\right|}{1+\frac{1+\alpha}{2}a_3}\le\frac{2\alpha}{1+\alpha} \qquad (5.9)$$

Let us multiply the inequalities (5.7) - (5.9), transpose the denominators and require that

$$\frac{\left|a_2+a_3-\frac{2}{1+\alpha}\right|}{1+\frac{1+\alpha}{2}a_2}\le\frac{2\alpha}{1+\alpha}$$

Remove moduli and add to the resulting inequality We obtain ($-\frac{2}{1+\alpha}$), where **$1\le\alpha$,**

$$\frac{\left|a_3-\frac{4}{1+\alpha}\right|}{1+\frac{1+\alpha}{2}a_2}\le\frac{2(\alpha-1)}{1+\alpha}$$

Analogously we get that

$$\frac{\left|a_2-\frac{4}{1+\alpha}\right|}{1+\frac{1+\alpha}{2}a_3}\le\frac{2(\alpha-1)}{1+\alpha},\qquad \frac{\left|a_1-\frac{4}{1+\alpha}\right|}{1+\frac{1+\alpha}{2}a_2}\le\frac{2(\alpha-1)}{1+\alpha}.$$

Multiply again the last three inequalition, transpose the denominaters and require that

$$\frac{\left|a_i - \dfrac{4}{1+\alpha}\right|}{1+\dfrac{1+\alpha}{2}a_i} \le \frac{2(\alpha-1)}{1+\alpha}, \qquad \left(i=\overline{1,3}\right)$$

Then we obtain that

$$2\frac{3-\alpha}{\alpha(1+\alpha)} \le a_i \le \frac{2}{2-\alpha} \qquad \text{, if } 1\le \boldsymbol{\alpha} \le \mathbf{2.}$$

Asymptotically , if $\dfrac{\tau}{h_i^2} \to \infty$, we have for ρ

$$\left|\rho\left(\frac{2}{2-\alpha}\right)\right| = \sqrt{\left(\frac{\left|1-\dfrac{5-\alpha}{2}a_i\right|}{1+\dfrac{1+\alpha}{2}a_i}\right)^3} = \sqrt{\left(\frac{\left|1-\dfrac{5-\alpha}{2}\cdot\dfrac{2}{2-\alpha}\right|}{1+\dfrac{1+\alpha}{2}\cdot\dfrac{2}{2-\alpha}}\right)^3} = 1.$$

For comparison, see [5], §2, the scheme (12).

Note that the transposition of the denominators required for (5.6) is equivalent to the following compotational algorithm in equation (4.7) for **k = 1,** the "sweeep" method is being used along the OX_1 axis, $i_1 = \overline{1, N_1}$ for fixed i_2 and i_3. When **k=2**, then $i_2 = \overline{1, N_2}$ for fixed i_1 and i_3. While when **k=3**, the "sweep" method is used along the axis OX_3, $i_3 = \overline{1, N_1}$, for fixed i_1 and i_2.

It have been considered also the cases with **p = 4,5,6,....** With thear help has been established the general rule for estimation of a_i.

First, let us write down the dispersion equation for any **p :**

$$\left(1+\frac{1+\alpha}{2}a_i\right)\rho^2 + \left(\frac{1-\alpha}{2}a_i + \sum_{\substack{k=1\\k\neq i}}^{p} a_k - 1\right) = 0, \qquad i=\overline{1,p}$$

We require that

$$|\rho| = |\rho_1\cdot\rho_2\cdots\rho_p| = \sqrt{\frac{\left|1-\dfrac{1-\alpha}{2}a_1 - a_2 - \ldots - a_p\right|}{1+\dfrac{1+\alpha}{2}a_1}\cdots\frac{\left|1-\dfrac{1-\alpha}{2}a_p - a_1 - a_2 - \ldots - a_{p-1}\right|}{1+\dfrac{1+\alpha}{2}a_p}} \le 1.$$

By means of this expression we obtain the following inequalities for the quanitities a_i.

$$0 < 2\frac{2p-(\alpha+3)}{(3+\alpha-p)(1+\alpha)} \le a_i \le \frac{2}{(p-1)-\alpha} \tag{5.10}$$

where **p-2 ≤ α≤ p –1 (p≥3).**

Asymptotically, i.e. when the value of $\dfrac{\tau}{h_i^2}$ is large enough, we get the following inequality for $\rho(\alpha)$

$$|\rho(p-1)| = \sqrt{\left(\frac{\left|1-\dfrac{2p-\alpha-1}{2}a\right|}{1+\dfrac{1+\alpha}{2}a}\right)^p} = \sqrt{\left(\frac{\left|1-\dfrac{p}{2}a\right|}{1+\dfrac{p}{2}a}\right)^p} \le 1$$

I.e. in general, for estimation (5.10), when **α=p-1**, we get unconditional condition of stability

$$0 < \frac{\tau}{h^2} < \infty \qquad \mathbf{p = 1,2, \ldots ,}$$

In the sequal, we exploit the following representation of scheme (4.7) :

$$\frac{u^{n+k}-u^{n+k-2}}{2\tau}=\Delta_{kk}\left(\frac{p}{2}u^{n+k}+\left(1-\frac{p}{2}\right)u^{n+k-2}\right)+\sum_{\substack{i=1\\ i\neq k}}^{p}\Delta_{ii}u^{n+k-2}+f^{n+k-1} \qquad (5.11)$$

$k=\overline{1,p},$ or well the fractional step notations

$$\frac{u^{n+\frac{k}{p}}-u^{n+\frac{k-1}{p}}}{\frac{1}{p}\tau}=0.5p\Delta_{kk}\left(u^{n+\frac{k}{p}}-u^{n+\frac{k-1}{p}}\right)+\sum_{i=1}^{p}\Delta_{ii}u^{n+\frac{k-1}{p}}+f^{n+\frac{2k-1}{2p}}, \qquad (5.12)$$

$k=\overline{1,p}$.

This permits us to make the following conclusion : the two-layer finite difference schemes (5.11), or which is the same, the schemes (5.12), represent a generalization of existing economic schemes for multidimensional parabolic equations with constant coefficients. Besides, the choice with weight α permited the schemes to be dependent only on the dimension **p**.

Let us schow that the scheme (5.12) approximates equation (1.1) and has exactness of second order with respect to steps. Let us consider first the case p=3. Analogously to [7] we write down scheme (5.12) in the following form :

$$A_1u^{n+\frac{1}{3}}=A_1u^n+Bu^n,\quad A_2u^{n+\frac{2}{3}}=A_2u^{n+\frac{1}{3}}+Bu^{n+\frac{1}{3}},\quad A_3u^{n+1}=A_3u^{n+\frac{2}{3}}+Bu^{n+\frac{2}{3}}$$

where $A_i=E-\frac{\tau}{2}\Delta_{ii}$, i=1,2,3. $B=\frac{\tau}{3}(\Delta_{11}+\Delta_{22}+\Delta_{33})$.

Excepting in turn $u^{n+\frac{1}{3}}$, $u^{n+\frac{2}{3}}$, we obtain the following equivalent scheme

$$A_1A_2A_3u^{n+1}=A_1A_2A_3u^n+\{B(A_1A_2+A_1A_3+A_2A_3)+B^2(A_1+A_2+A_3)+B^3\}u^n$$

If we insert the values of operators $\mathbf{A_j}$, **B** and represent with respect to τ in the expanded form, obtain

$$\frac{u^{n+1}-u^n}{\tau}=(\Delta_{11}+\Delta_{22}+\Delta_{33})\frac{u^{n+1}+u^n}{2}-\left(\frac{\tau}{2}\right)^2(\Delta_{11}\Delta_{22}+\Delta_{11}\Delta_{33}+\Delta_{22}\Delta_{33})\frac{u^{n+1}-u^n}{2}+$$

$$+\left(\frac{\tau}{2}\right)^3\Delta_{11}\Delta_{22}\Delta_{33}\frac{u^{n+1}-u^n}{\tau}+\tau^2\left\{\frac{1}{12}(\Delta_{11}+\Delta_{22}+\Delta_{33})(\Delta_{11}\Delta_{22}+\Delta_{11}\Delta_{33}+\Delta_{22}\Delta_{33})-\right.$$

$$\left.-\frac{1}{54}(\Delta_{11}+\Delta_{22}+\Delta_{33})^3\right\}u^n$$

This scheme is equivalent to the scheme

$$\frac{u^{n+1}-u^n}{\tau}=(\Delta_{11}+\Delta_{22}+\Delta_{33})\frac{u^{n+1}+u^n}{2}$$

and approximates the heat conductivity equation with the exactness $\mathbf{O(\tau^{2}+|h|^{2})}$.

Analogouly we can consider the case of arbitrary p **(p>3).** Let us write down scheme (5.12) in the following form

$$A_1u^{n+\frac{1}{p}}=(A_1+B)u^n$$

$$A_2u^{n+\frac{2}{p}}=(A_2+B)u^{n+\frac{1}{p}}$$

$$-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-\;-$$

$$A_pu^{n+1}=(A_p+B)u^{n+\frac{p-1}{p}}$$

where $A_i = E - \frac{\tau}{2}\Delta_{ii}$, $i = \overline{1,p}$, $B = \frac{\tau}{p}\sum_{i=1}^{p}\Delta_{ii}$.

Excepting in turn $u^{n+\frac{1}{p}}$ ... $u^{n+\frac{p-1}{p}}$, we obtain

$$A_1 A_2 ...A_p u^{n+1} = A_1 A_2 ...A_p u^n + \{B(A_2 A_3 ...A_p+...+A_1 A_2 ...A_{p-2} A_{p-1}) +$$
$$+B^2(A_3 A_4 ...A_p+...+A_1 A_2 ...A_{p-3} A_{p-2}) + ... +B^{p-1}(A_1 +A_2 +...+A_p) +B^p\} u^n$$

If we tepresent in the expanded form with respect to τ , obtain

$$\frac{u^{n+1}-u^n}{\tau} = (\Delta_{11}+\cdots+\Delta_{pp})\frac{u^n+u^{n+1}}{2} - \left(\frac{\tau}{2}\right)^2(\Delta_{11}\Delta_{22}\cdots\Delta_{p-1p-1}+\cdots+\Delta_{11}\Delta_{22}\cdots\Delta_{p-2p-2}\cdot\Delta_{pp})\cdot$$
$$\cdot\frac{u^{n+1}-u^n}{\tau}+\cdots+\left(\frac{\tau}{p}\right)^2\Delta_{11}\Delta_{22}\cdots\Delta_{pp}\frac{u^{n+1}-u^n}{\tau}+0(\tau^2)$$

From this follows that the last is equivalent to scheme (5.12) and approximates the heat conductivity equation (1.1) with the same exactness as the following scheme

$$\frac{u^{n+1}-u^n}{\tau} = (\Delta_{11}+\cdots+\Delta_{pp})\frac{u^{n+1}+u^n}{2}.$$

6. Some remarks on the step-like schemes

(5.11) and fractical step (see [3], [5]).

1. Write down the equivalent schemes (5.11) and (5.12) when **p = 1**

$$\frac{u^{n+1}-u^{n-1}}{2\tau} = \frac{1}{2}\Delta_{11}\left(u^{n+1}+u^{n-1}\right)+f^n$$

$$\frac{u^{n+1}-u^n}{\tau} = \frac{1}{2}\Delta_{11}\left(u^{n+1}+u^n\right)+f^{n+\frac{1}{2}}$$

if **f = 0,** we get the Krank-Nicolson scheme (see [5], § 2, p.8).

2. Analogously consider the case with **p = 2**

$$\begin{cases}\dfrac{u^{n+1}-u^{n-1}}{2\tau} = \Delta_{11}u^{n+1}+\Delta_{22}u^{n-1}+f^n\\[2ex] \dfrac{u^{n+2}-u^n}{2\tau} = \Delta_{22}u^{n+2}+\Delta_{11}u^n+f^{n+1}\end{cases};\quad \begin{cases}\dfrac{u^{n+\frac{1}{2}}-u^n}{0{,}5\tau} = \Delta_{11}u^{n+\frac{1}{2}}+\Delta_{22}u^n+f^{n+\frac{1}{4}}\\[2ex] \dfrac{u^{n+1}-u^{n+\frac{1}{2}}}{0{,}5\tau} = \Delta_{22}u^{n+1}+\Delta_{11}u^{n+\frac{1}{2}}+f^{n+\frac{3}{4}}\end{cases} \qquad (6.1)$$

If **f = 0**, (6.1) represents the schemes of Pismen, Rochford and Douglas, see [1-2]. Generally, if **f ≠ 0**, the difference between the schemes (5.11) and splitted or fractional-step schemes (the latter is equivalent to factorized schemes concordant to the non-zero boundary conditions see [3], IX, §1) consists just in calculation of **f.** This seems to be natural, since (5.11) schemes, unlike to other economical schemes, completely approximate the given differential equation.

3. For the finite-difference scheme (5.12) non lomogenorsl boundary conditions u(x,t)= $\varphi(x,t)$, (x,t)єΓ , can implemented as folloms: $u^{n+\frac{k}{p}}(x) = \varphi\left(x,\left(n+\frac{k}{p}\right)\tau\right)$, $x \in \Gamma$

$$u^{n+1}(x) = \varphi(x,(n+1)\tau) \ , \quad x \in \Gamma$$

Notice that in this scheme maintains second order accuruey , mhile to achieve the same effect with factored numerical schemes special boundary formulees must be lerived for $u^{n+\frac{k}{p}}$ see e.g. [3, appendix,632].

4. The continuity of solution of difference problem with respect to the initial conditions and to the right hand side of equation, follows form stability of scheme (5.11).

Constructed by the suggested method three-layer schemes for the second order parabolic and hyperbolic equations with variable coefficients, see [8,9] whose investigetion are based on the general principle of regularization were tested repeatedly by a lot of tests. The case of three layer parabolic equations we apply our method for one practical specific boundary problem of hydrodynamic and its numerical solutions are represented in a tabular form (see [10]).

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სხვაობიანი სქემები მრავალგანზომილებიანი, მუდმივკოეფიციენტებიანი პარაბოლური ტიპის განტოლების ინტეგრებისათვის

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რეზიუმე

განიხილება მრავალგანზომილებიანი პარაბოლური ტიპის განტოლების ინტეგრებისათვის მარტივი, სიმეტრიული, აბსოლუტურად მდგრადი წილად-ბიჯიანი სქემების აგების ალგორითმი. აღსანიშნავია, რომ თითოეული სხვაობიანი განტოლება სრულად ააპროქსიმირებს მოცემულ დიფერენციალურ განტოლებას. სქემების მდგრადობის გამოკვლევისათვის გამოიყენება ჰარმონიული ანალიზის მეთოდი.

РАЗНОСТНЫЕ СХЕМЫ ДЛЯ ИНТЕГРИРОВАНИЯ МНОГОМЕРНЫХ УРАВНЕНИЙ ПАРАБОЛИЧЕСКОГО ТИПА С ПОСТОЯННЫМИ КОЭФФИЦИЕНТАМИ

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Резюме

Рассмотривается алгоритм построения простых симметризованых абсолютного устойчивых дробно-шаговых схем, для многомерного уравнения параболического типа. Для исследования устойчивости разностных схем используется метод гармонического анализа.